6 Continuous Probability Distributions

CHAPTER CONTENTS

LEARNING OBJECTIVES

After reading this chapter and doing the exercises, you should be able to:

1. Understand the difference between how probabilities are computed for discrete and continuous random variables.
2. Compute probability values for a continuous uniform probability distribution and be able to compute the expected value and variance for such a distribution.
3. Compute probabilities using a normal probability distribution. Understand the role of the standard normal distribution in this process.
4. Use the normal distribution to approximate binomial probabilities.
5. Compute probabilities using an exponential probability distribution.
6. Understand the relationship between the Poisson and exponential probability distributions.

In this chapter we turn to the study of continuous random variables. Specifically, we discuss three continuous probability distributions: the uniform, the normal and the exponential. A fundamental difference separates discrete and continuous random variables in terms of how probabilities are computed. For a discrete random variable, the probability function $p(x)$ provides the probability that the random variable assumes a particular value. With continuous random variables the counterpart of the probability function is the **probability density function**, denoted by $f(x)$. The difference is that the probability density function does not directly provide probabilities. However, the area under the graph of $f(x)$ corresponding to a given interval does provide the probability that the continuous random variable

X assumes a value in that interval. So when we compute probabilities for continuous random variables we are computing the probability that the random variable assumes any value in an interval.

One of the implications of the definition of probability for continuous random variables is that the probability of any particular value of the random variable is zero, because the area under the graph of $f(x)$ at any particular point is zero. In Section 6.1 we demonstrate these concepts for a continuous random variable that has a uniform distribution.

Much of the chapter is devoted to describing and showing applications of the normal distribution. The main importance of normal distribution is its extensive use in statistical inference. The chapter closes with a discussion of the exponential distribution.

STATISTICS IN PRACTICE

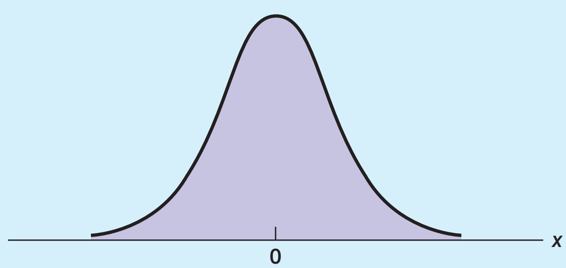
Assessing the effectiveness of new medical procedures

Clinical trials are a vital and commercially very important application of statistics, typically involving the random assignment of patients to two experimental groups. One group receives the treatment of interest, the second a placebo (a dummy treatment that has no effect). To assess the evidence that the probability of success with the treatment will be better than that with the placebo, frequencies a, b, c and d can be collected for a predetermined number of trials according to the following two-way table:

	Treatment	*Placebo*
Success	a	b
Failure	c	d

and the quantity ('log odds ratio') $X = \log(a/c/b/d)$ calculated. Clearly the larger the value of X obtained the greater the evidence that the treatment is better than the placebo.

In the particular case that the treatment has no effect, the distribution of X can be shown to align very closely to a normal distribution with a mean of zero:



Thus, as values of X fall increasingly to the right of the zero mean this should signify stronger and stronger support for the belief in the treatment's relative effectiveness.

Intriguingly, this formulation was adapted by Copas (2005) to cast doubt on the findings of a recent study linking passive smoking to an increased risk of lung cancer.

A participant takes part in a new drugs trial

Source: Copas, John (2005) 'The downside of publication'. *Significance* Vol. 2 Issue 4 pp. 154–157.

6.1 UNIFORM PROBABILITY DISTRIBUTION

Consider the random variable X representing the flight time of an aeroplane travelling from Graz to Stansted. Suppose the flight time can be any value in the interval from 120 minutes to 140 minutes. Because the random variable X can assume any value in that interval, X is a continuous rather than a discrete random variable. Let us assume that sufficient actual flight data are available to conclude that the probability of a flight time within any one-minute interval is the same as the probability of a flight time within any other one-minute interval contained in the larger interval from 120 to 140 minutes. With every one-minute interval being equally likely, the random variable X is said to have a **uniform probability distribution**.

If x is any number lying in the range that the random variable X can take then the probability density function, which defines the uniform distribution for the flight-time random variable, is:

$$f(x) = \begin{cases} 1/20 & \text{for } 120 \leq x \leq 140 \\ 0 & \text{elsewhere} \end{cases}$$

Figure 6.1 is a graph of this probability density function. In general, the uniform probability density function for a random variable X is defined by the following formula.

Uniform probability density function

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \leq x \leq b. \\ 0 & \text{elsewhere} \end{cases} \qquad \textbf{(6.1)}$$

For the flight-time random variable, $a = 120$ and $b = 140$.

As noted in the introduction, for a continuous random variable, we consider probability only in terms of the likelihood that a random variable assumes a value within a specified interval. In the flight time example, an acceptable probability question is: What is the probability that the flight time is between 120 and 130 minutes? That is, what is $P(120 \leq X \leq 130)$? Because the flight time must be between 120 and 140 minutes and because the probability is described as being uniform over this interval, we feel comfortable saying $P(120 \leq X \leq 130) = 0.50$. In the following subsection we show that this probability can be computed as the area under the graph of $f(x)$ from 120 to 130 (see Figure 6.2).

FIGURE 6.1
Uniform probability density function for flight time

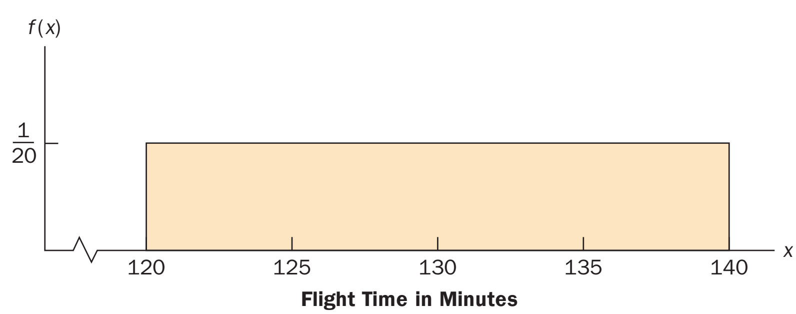
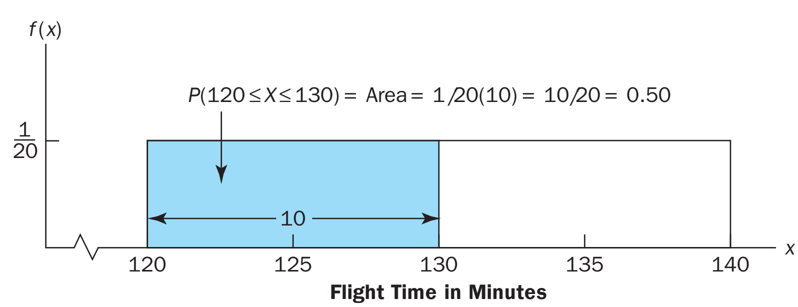


FIGURE 6.2
Area provides probability of flight time between 120 and 130 minutes



Area as a measure of probability

Let us make an observation about the graph in Figure 6.2. Consider the area under the graph of $f(x)$ in the interval from 120 to 130. The area is rectangular, and the area of a rectangle is simply the width multiplied by the height. With the width of the interval equal to $130 - 120 = 10$ and the height equal to the value of the probability density function $f(x) = 1/20$, we have area = width $\times$ height = $10 \times 1/20 = 10/20 = 0.50$.

What observation can you make about the area under the graph of $f(x)$ and probability? They are identical! Indeed, this observation is valid for all continuous random variables. Once a probability density function $f(x)$ is identified, the probability that X takes a value x between some lower value x_1 and some higher value x_2 can be found by computing the area under the graph of $f(x)$ over the interval from x_1 to x_2.

Given the uniform distribution for flight time and using the interpretation of area as probability, we can answer any number of probability questions about flight times. For example, what is the probability of a flight time between 128 and 136 minutes? The width of the interval is $136 - 128 = 8$. With the uniform height of $f(x) = 1/20$, we see that $P(128 \leq X \leq 136) = 8 \times 1/20 = 0.40$. Note that $P(120 \leq X \leq 140) = 20 \times 1/20 = 1$; that is, the total area under the graph of $f(x)$ is equal to 1. This property holds for all continuous probability distributions and is the analogue of the condition that the sum of the probabilities must equal 1 for a discrete probability function. For a continuous probability density function, we must also require that $f(x) \geq 0$ for all values of x. This requirement is the analogue of the requirement that $p(x) \geq 0$ for discrete probability functions.

Two major differences stand out between the treatment of continuous random variables and the treatment of their discrete counterparts.

1. We no longer talk about the probability of the random variable assuming a particular value. Instead, we talk about the probability of the random variable assuming a value within some given interval.
2. The probability of the random variable assuming a value within some given interval from x_1 to x_2 is defined to be the area under the graph of the probability density function between x_1 and x_2. It implies that the probability of a continuous random variable assuming any particular value exactly is zero, because the area under the graph of $f(x)$ at a single point is zero.

The calculation of the expected value and variance for a continuous random variable is analogous to that for a discrete random variable. However, because the computational procedure involves integral calculus, we leave the derivation of the appropriate formulae to more advanced texts.

For the uniform continuous probability distribution introduced in this section, the formulae for the expected value and variance are:

$$E(X) = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

In these formulae, a is the smallest value and b is the largest value that the random variable may assume.

Applying these formulae to the uniform distribution for flight times from Graz to Stansted, we obtain:

$$E(X) = \frac{(120+140)}{2} = 130$$

$$\mathrm{Var}(X) = \frac{(140-120)^2}{12} = 33.33$$

The standard deviation of flight times can be found by taking the square root of the variance. Thus, $\sigma = 5.77$ minutes.

EXERCISES

Methods

1. The random variable X is known to be uniformly distributed between 1.0 and 1.5.

- **a.** Show the graph of the probability density function.
- **b.** Compute $P(X = 1.25)$.
- **c.** Compute $P(1.0 \leq X \leq 1.25)$.
- **d.** Compute $P(1.20 < X < 1.5)$.

2. The random variable X is known to be uniformly distributed between 10 and 20.

- **a.** Show the graph of the probability density function.
- **b.** Compute $P(X < 15)$.
- **c.** Compute $P(12 \leq X \leq 18)$.
- **d.** Compute $E(X)$.
- **e.** Compute $\mathrm{Var}(X)$.

COMPLETE SOLUTIONS

Applications

3. A continuous random variable X has probability density function:

$$\begin{array}{ll} f(x) = kx & 0 < x < 2 \\ 0 \;\; \textit{otherwise} & \end{array}$$

- **a.** Determine the value of k.
- **b.** Find $E(X)$ and $Var(X)$.
- **c.** What is the probability that X is greater than three standard deviations above the mean?
- **d.** Find the distribution function $F(X)$ and hence the median of X.

4. Most computer languages include a function that can be used to generate random numbers. In EXCEL, the RAND function can be used to generate random numbers between 0 and 1. If we let X denote a random number generated using RAND, then X is a continuous random variable with the following probability density function.

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

- **a.** Graph the probability density function.
- **b.** What is the probability of generating a random number between 0.25 and 0.75?
- **c.** What is the probability of generating a random number with a value less than or equal to 0.30?
- **d.** What is the probability of generating a random number with a value greater than 0.60?

5. Let X denote the number of bricks a bricklayer will lay in an hour and assume that X takes values in the range 150 to 200 inclusively with equal probability (i.e. has a discrete uniform distribution). If a certain project is 170 bricks short of completion and a further project is waiting to be started as soon as this one is finished, what is the probability that:

- **a.** The bricklayer will start the second project within the hour?
- **b.** More than 25 bricks will have been laid on the second project at the end of the next hour?
- **c.** The first project will be more than ten bricks short of completion at the end of the next hour?
- **d.** The bricklayer will lay exactly 175 bricks during the next hour?

6. The label on a bottle of liquid detergent shows contents to be 12 grams per bottle. The production operation fills the bottle uniformly according to the following probability density function.

$$f(x) = \begin{cases} 8 & \text{for } 11.975 \leq x \leq 12.100 \\ 0 & \text{elsewhere} \end{cases}$$

- **a.** What is the probability that a bottle will be filled with between 12 and 12.05 grams?
- **b.** What is the probability that a bottle will be filled with 12.02 or more grams?
- **c.** Quality control accepts a bottle that is filled to within 0.02 grams of the number of grams shown on the container label. What is the probability that a bottle of this liquid detergent will fail to meet the quality control standard?

7. Suppose we are interested in bidding on a piece of land and we know there is one other bidder. The seller announced that the highest bid in excess of €10 000 will be accepted. Assume that the competitor's bid X is a random variable that is uniformly distributed between €10 000 and €15 000.

- **a.** Suppose you bid €12 000. What is the probability that your bid will be accepted?
- **b.** Suppose you bid €14 000. What is the probability that your bid will be accepted?
- **c.** What amount should you bid to maximize the probability that you get the property?
- **d.** Suppose you know someone who is willing to pay you €16 000 for the property. Would you consider bidding less than the amount in part (c)? Why or why not?

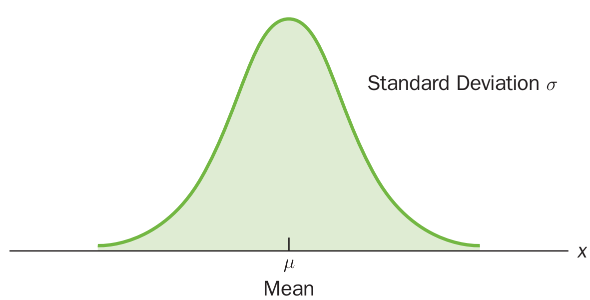
6.2 NORMAL PROBABILITY DISTRIBUTION

The most important probability distribution for describing a continuous random variable is the **normal probability distribution**. The normal distribution has been used in a wide variety of practical applications in which the random variables are heights and weights of people, test scores, scientific measurements, amounts of rainfall and so on. It is also widely used in statistical inference, which is the major topic of the remainder of this book. In such applications, the normal distribution provides a description of the likely results obtained through sampling.

Normal curve

The form, or shape, of the normal distribution is illustrated by the bell-shaped normal curve in Figure 6.3. The probability density function that defines the bell-shaped curve of the normal distribution follows.

FIGURE 6.3
Bell-shaped curve for the normal distribution



Normal probability density function

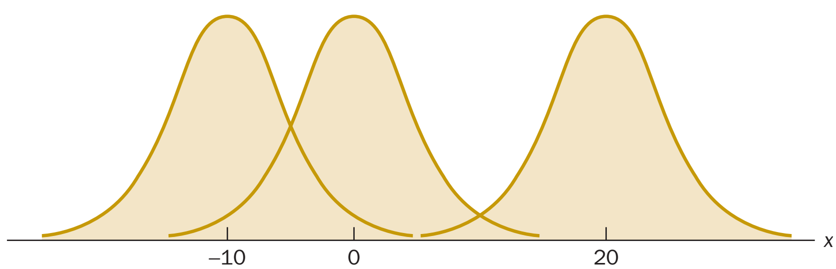
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)1/2\sigma^2} \quad \textbf{(6.2)}$$

where

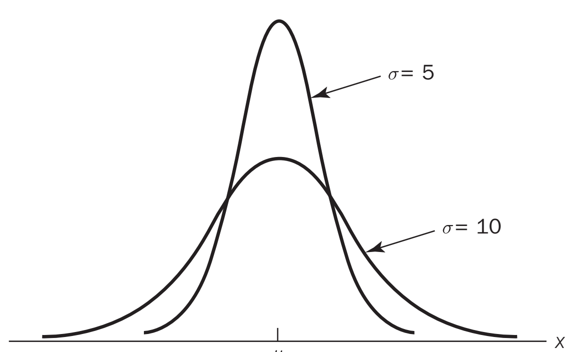
μ = mean
σ = standard deviation
π = 3.14159
e = 2.71828

We make several observations about the characteristics of the normal distribution:

1. The entire family of normal distributions is differentiated by its mean μ and its standard deviation σ.
2. The highest point on the normal curve is at the mean, which is also the median and mode of the distribution.
3. The mean of the distribution can be any numerical value: negative, zero or positive. Three normal distributions with the same standard deviation but three different means (−10, 0 and 20) are shown here.



4. The normal distribution is symmetric, with the shape of the curve to the left of the mean a mirror image of the shape of the curve to the right of the mean. The tails of the curve extend to infinity in both directions and theoretically never touch the horizontal axis. Because it is symmetric, the normal distribution is not skewed; its skewness measure is zero.
5. The standard deviation determines how flat and wide the curve is. Larger values of the standard deviation result in wider, flatter curves, showing more variability in the data. Two normal distributions with the same mean but with different standard deviations are shown here.



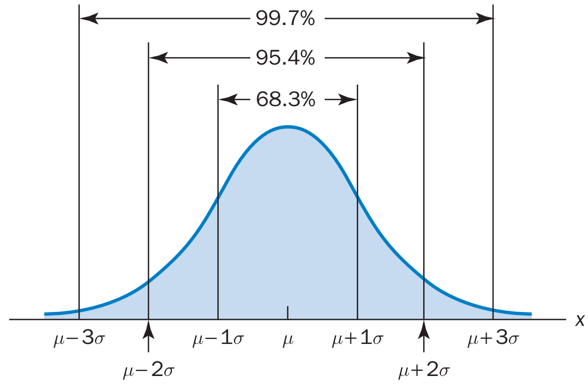


FIGURE 6.4
Areas under the curve for any normal distribution

6 Probabilities for the normal random variable are given by areas under the curve. The total area under the curve for the normal distribution is 1. Because the distribution is symmetric, the area under the curve to the left of the mean is 0.50 and the area under the curve to the right of the mean is 0.50.

7 The percentage of values in some commonly used intervals are:

a. 68.3 per cent of the values of a normal random variable are within plus or minus one standard deviation of its mean.
b. 95.4 per cent of the values of a normal random variable are within plus or minus two standard deviations of its mean.
c. 99.7 per cent of the values of a normal random variable are within plus or minus three standard deviations of its mean.

Figure 6.4 shows properties (a), (b) and (c) graphically.

Standard normal probability distribution

A random variable that has a normal distribution with a mean of zero and a standard deviation of one is said to have a **standard normal probability distribution**. The letter Z is commonly used to designate this particular normal random variable. Figure 6.5 is the graph of the standard normal distribution. It has the same general appearance as other normal distributions, but with the special properties of $\mu = 0$ and $\sigma = 1$.

Standard normal density function

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-x^{1/2}}$$

Because $\mu = 0$ and $\sigma = 1$, the formula for the standard normal probability density function is a simpler version of equation (6.2).

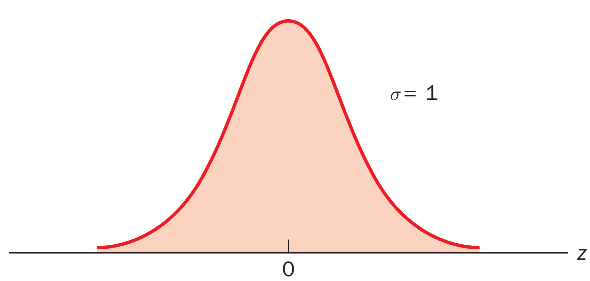


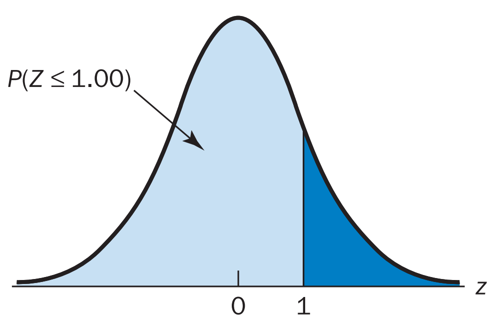
FIGURE 6.5
The standard normal distribution

As with other continuous random variables, probability calculations with any normal distribution are made by computing areas under the graph of the probability density function. Thus, to find the probability that a normal random variable is within any specific interval, we must compute the area under the normal curve over that interval.

For the standard normal distribution, areas under the normal curve have been computed and are available in tables that can be used to compute probabilities. Such a table appears on the two pages inside the front cover of the text. The table on the left-hand page contains areas, or cumulative probabilities, for z values less than or equal to the mean of zero. The table on the right-hand page contains areas, or cumulative probabilities, for z values greater than or equal to the mean of zero.

The three types of probabilities we need to compute include (1) the probability that the standard normal random variable Z will be less than or equal to a given value; (2) the probability that Z will take a value between two given values; and (3) the probability that Z will be greater than or equal to a given value. To see how the cumulative probability table for the standard normal distribution can be used to compute these three types of probabilities, let us consider some examples.

We start by showing how to compute the probability that Z is less than or equal to 1.00; that is, $P(Z \leq 1.00)$. This cumulative probability is the area under the normal curve to the left of $z = 1.00$ in the following graph.

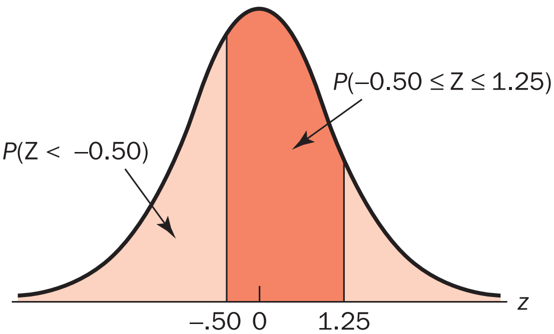


Refer to the right-hand page of the standard normal probability table inside the front cover of the text. The cumulative probability corresponding to $z = 1.00$ is the table value located at the intersection of the row labelled 1.0 and the column labelled .00. First we find 1.0 in the left column of the table and then find .00 in the top row of the table. By looking in the body of the table, we find that the 1.0 row and the .00 column intersect at the value of 0.8413; thus, $P(Z \leq 1.00) = 0.8413$. The following excerpt from the probability table shows these steps.

z	.00	.01	.02
.			
.			
.			
.9	.8159	.8186	.8212
1.0	$\boxed{.8413}$	.8438	.8461
1.1	.8643	.8665	.8686
1.2	.8849	.8869	.8888
.			
.			
.			

$P(Z \leq 1.00)$

To illustrate the second type of probability calculation we show how to compute the probability that Z is in the interval between -0.50 and 1.25; that is, $P(-0.50 \leq Z \leq 1.25)$. The following graph shows this area, or probability.

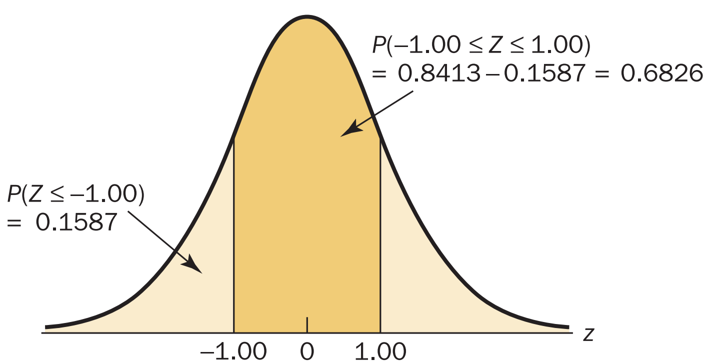


Three steps are required to compute this probability. First, we find the area under the normal curve to the left of $z = 1.25$. Second, we find the area under the normal curve to the left of $z = -0.50$. Finally, we subtract the area to the left of $z = -0.50$ from the area to the left of $z = 1$ to find $P(-0.5 \leq Z \leq 1.25)$.

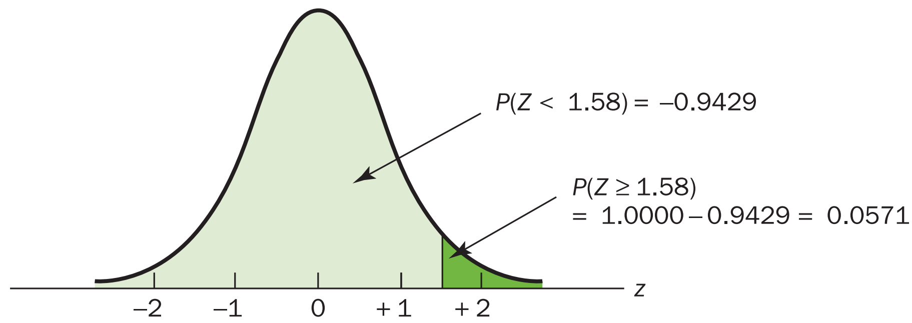
To find the area under the normal curve to the left of $z = 1.25$, we first locate the 1.2 row in the standard normal probability table and then move across to the .05 column. Because the table value in the 1.2 row and the .05 column is 0.8944, $P(Z \leq 1.25) = 0.8944$. Similarly, to find the area under the curve to the left of $z = -0.50$ we use the left-hand page of the table to locate the table value in the -0.5 row and the .00 column; with a table value of 0.3085, $P(Z \leq -0.50) = 0.3085$. Thus, $P(-0.50 \leq Z \leq 1.25) = P(Z \leq 1.25) - P(Z \leq -0.50) = 0.8944 - 0.3085 = 0.5859$.

Let us consider another example of computing the probability that Z is in the interval between two given values. Often it is of interest to compute the probability that a normal random variable assumes a value within a certain number of standard deviations of the mean. Suppose we want to compute the probability that the standard normal random variable is within one standard deviation of the mean; that is, $P(-1.00 \leq Z \leq 1.0)$.

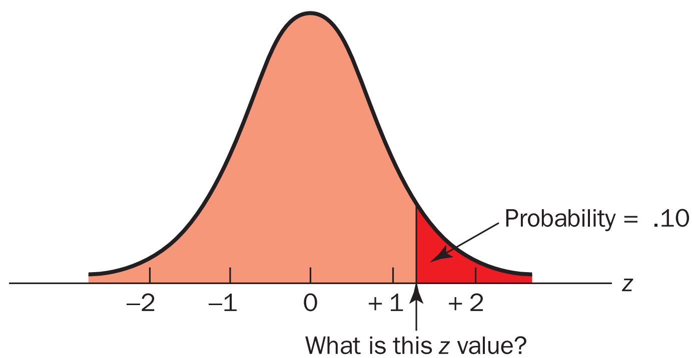
To compute this probability we must find the area under the curve between -1.0 and 1.00. Earlier we found that $P(Z \leq 1.00) = 0.8413$. Referring again to the table inside the front cover of the book, we find that the area under the curve to the left of $z = -1.00$ is 0.1587, so $P(Z \leq -1.00) = 0.1587$. Therefore $P(-1.00 \leq Z \leq 1.00) = P(Z \leq 1.00) - P(Z \leq -1.00) = 0.8413 - 0.1587 = 0.6826$. This probability is shown graphically in the following figure.



To illustrate how to make the third type of probability computation, suppose we want to compute the probability of obtaining a z value of at least 1.58; that is, $P(Z \geq 1.58)$. The value in the $z = 1.5$ row and the .08 column of the cumulative normal table is 0.9429; thus, $P(Z < 1.58) = 0.9429$. However, because the total area under the normal curve is 1, $P(Z \geq 1.58) = 1 - 0.9429 = 0.0571$. This probability is shown in the following figure.



In the preceding illustrations, we showed how to compute probabilities given specified z values. In some situations, we are given a probability and are interested in working backward to find the corresponding z value. Suppose we want to find a z value such that the probability of obtaining a larger z value is 0.10. The following figure shows this situation graphically.



This problem is the inverse of those in the preceding examples. Previously, we specified the z value of interest and then found the corresponding probability, or area. In this example, we are given the probability, or area, and asked to find the corresponding z value. To do so, we use the standard normal probability table somewhat differently.

z	.06	.07	.08	.09
.				
.				
.				
1.0	.8554	.8577	.8599	.8621
1.1	.8770	.8790	.8810	.8830
1.2	.8962	.8980	$\boxed{.8997}$	.9015
1.3	.9131	.9147	.9162	.9177
1.4	.9279	.9292	.9306	.9319
.				
.				
.				

Cumulative probability value
closest to 0.9000

Recall that the standard normal probability table gives the area under the curve to the left of a particular z value. We have been given the information that the area in the upper tail of the curve is 0.10. Hence, the area under the curve to the left of the unknown z value must equal 0.9000. Scanning the body of the table, we find 0.8997 is the cumulative probability value closest to 0.9000. The section of the table providing this result is shown above. Reading the z value from the left-most column and the top row of the table, we find that the corresponding z value is 1.28. Thus, an area of approximately 0.9000 (actually

0.8997) will be to the left of $z = 1.28$.* In terms of the question originally asked, the probability is approximately 0.10 that the z value will be larger than 1.28.*

The examples illustrate that the table of areas for the standard normal distribution can be used to find probabilities associated with values of the standard normal random variable Z. Two types of questions can be asked. The first type of question specifies a value, or values, for z and asks us to use the table to determine the corresponding areas, or probabilities.

The second type of question provides an area, or probability, and asks us to use the table to determine the corresponding z value. Thus, we need to be flexible in using the standard normal probability table to answer the desired probability question. In most cases, sketching a graph of the standard normal distribution and shading the appropriate area or probability helps to visualize the situation and aids in determining the correct answer.

Computing probabilities for any normal distribution

The reason for discussing the standard normal distribution so extensively is that probabilities for all normal distributions are computed by using the standard normal distribution. That is, when we have a normal distribution with any mean μ and any standard deviation σ, we answer probability questions about the distribution by first converting to the standard normal distribution. Then we can use the standard normal probability table and the appropriate z values to find the desired probabilities. The formula used to convert any normal random variable X with mean μ and standard deviation σ to the standard normal distribution follows as equation (6.3).

Converting to the standard normal distribution

$$Z = \frac{X - \mu}{\sigma} \tag{6.3}$$

A value of X equal to the mean μ results in $z = (\mu - \mu)/\sigma = 0$. Thus, we see that a value of X equal to the mean μ *of* X corresponds to a value of Z at the mean 0 of Z. Now suppose that x is one standard deviation greater than the mean; that is, $x = \mu + \sigma$. Applying equation (6.3), we see that the corresponding z value $= [(\mu + \sigma) - \mu]/\sigma = \sigma/\sigma = 1$. Thus, a value of X that is one standard deviation above the mean μ *of* X corresponds to a z value $= 1$. In other words, we can interpret Z as the number of standard deviations that the normal random variable X is from its mean μ.

To see how this conversion enables us to compute probabilities for any normal distribution, suppose we have a normal distribution with $\mu = 10$ and $\sigma = 2$. What is the probability that the random variable X is between 10 and 14? Using equation (6.3) we see that at $x = 10$, $z = (x - \mu)/\sigma = (10 - 10)/2 = 0$ and that at $x = 14$, $z = (14 - 10)/2 = 4/2 = 2$. Thus, the answer to our question about the probability of X being between 10 and 14 is given by the equivalent probability that Z is between 0 and 2 for the standard normal distribution.

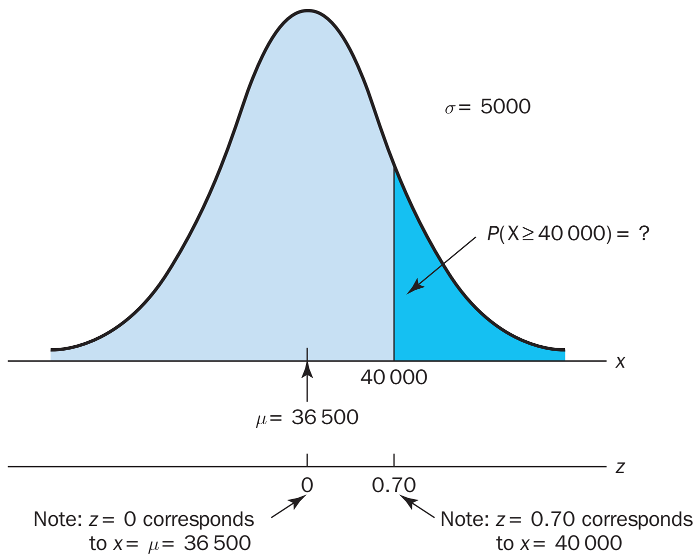
In other words, the probability that we are seeking is the probability that the random variable X is between its mean and two standard deviations greater than the mean. Using $z = 2.00$ and standard normal probability table, we see that $P(Z \leq 2) = 0.9772$. Because $P(Z \leq 0) = 0.5000$ we can compute $P(0.00 \leq Z \leq 2.00) = P(Z \leq 2) - P(Z \leq 0) = 0.9772 - 0.5000 = 0.4772$. Hence the probability that X is between 10 and 14 is 0.4772.

Greer Tyre Company problem

We turn now to an application of the normal distribution. Suppose the Greer Tyre Company just developed a new steel-belted radial tyre that will be sold through a national chain of discount stores. Because the tyre is a new product, Greer's managers believe that the kilometres guarantee offered with the tyre will be an

* We could use interpolation in the body of the table to get a better approximation of the z value that corresponds to an area of 0.9000. Doing so provides one more decimal place of accuracy and yields a z value of 1.282. However, in most practical situations, sufficient accuracy is obtained by simply using the table value closest to the desired probability.

FIGURE 6.6
Greer Tyre Company kilometres distribution



important factor in the acceptance of the product. Before finalizing the kilometres guarantee policy, Greer's managers want probability information about the number of kilometres the tyres will last.

From actual road tests with the tyres, Greer's engineering group estimates the mean number of kilometres the tyre will last is $\mu = 36\,500$ kilometres and that the standard deviation is $\sigma = 5000$. In addition, the data collected indicate a normal distribution is a reasonable assumption. What percentage of the tyres can be expected to last more than 40 000 kilometres?

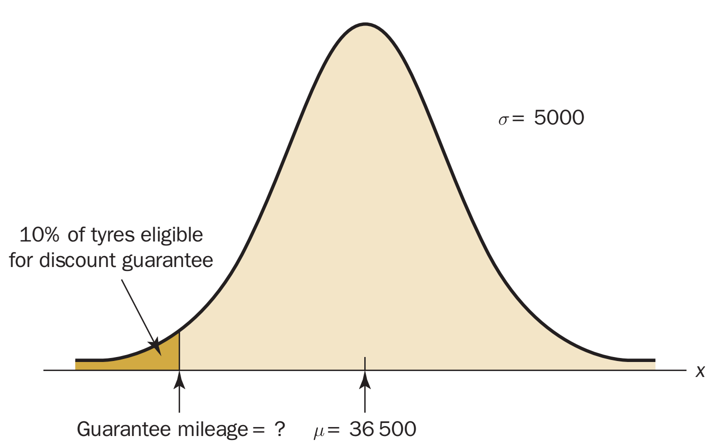
In other words, what is the probability that the number of kilometres the tyre lasts will exceed 40 000? This question can be answered by finding the area of the darkly shaded region in Figure 6.6. At $x = 40\,000$, we have

$$Z = \frac{X-\mu}{\sigma} = \frac{40\,000 - 36\,500}{5000} = \frac{3500}{5000} = 0.70$$

Refer now to the bottom of Figure 6.6. We see that a value of $x = 40\,000$ on the Greer Tyre normal distribution corresponds to a value of $z = 0.70$ on the standard normal distribution. Using the standard normal probability table, we see that the area to the left of $z = 0.70$ is 0.7580. Referring again to Figure 6.6, we see that the area to the left of $x = 40\,000$ on the Greer Tyre normal distribution is the same. Thus, $1.000 - 0.7580 = 0.2420$ is the probability that X will exceed 40 000. We can conclude that about 24.2 per cent of the tyres will last longer than 40 000 kilometres.

Let us now assume that Greer is considering a guarantee that will provide a discount on replacement tyres if the original tyres do not exceed the number of kilometres stated in the guarantee. What should the guaranteed number of kilometres be if Greer wants no more than 10 per cent of the tyres to be eligible for the discount guarantee? This question is interpreted graphically in Figure 6.7.

FIGURE 6.7
Greer's discount guarantee



According to Figure 6.7, the area under the curve to the left of the unknown guaranteed number of kilometers must be 0.10. So we must find the z value that cuts off an area of 0.10 in the left tail of a standard normal distribution. Using the standard normal probability table, we see that $z = -1.28$ cuts off an area of 0.10 in the lower tail.

Hence $z = -1.28$ is the value of the standard normal variable corresponding to the desired number of kilometres guarantee on the Greer Tyre normal distribution. To find the value of X corresponding to $z = -1.28$, we have:

$$z = \frac{x - \mu}{\sigma} = -1.28$$

$$x - \mu = -1.28\sigma$$

$$x = \mu - 1.28\sigma$$

With $\mu = 36\ 500$ and $\sigma = 5000$,

$$x = 36\,500 - 1.28 \times 5000 = 30\,100$$

Thus, a guarantee of 30 100 kilometres will meet the requirement that approximately 10 per cent of the tyres will be eligible for the guarantee. Perhaps, with this information, the firm will set its tyre kilometres guarantee at 30 000 kilometres.

Again, we see the important role that probability distributions play in providing decision-making information. Namely, once a probability distribution is established for a particular application, it can be used quickly and easily to obtain probability information about the problem. Probability does not establish a decision recommendation directly, but it provides information that helps the decision-maker better understand the risks and uncertainties associated with the problem. Ultimately, this information may assist the decision-maker in reaching a good decision.

EXERCISES

Methods

8. Using Figure 6.4 as a guide, sketch a normal curve for a random variable X that has a mean of $\mu = 100$ and a standard deviation of $\sigma = 10$. Label the horizontal axis with values of 70, 80, 90, 100, 110, 120 and 130.

9. A random variable is normally distributed with a mean of $\mu = 50$ and a standard deviation of $\sigma = 5$.

- **a.** Sketch a normal curve for the probability density function. Label the horizontal axis with values of 35, 40, 45, 50, 55, 60 and 65. Figure 6.4 shows that the normal curve almost touches the horizontal axis at three standard deviations below and at three standard deviations above the mean (in this case at 35 and 65).
- **b.** What is the probability the random variable will assume a value between 45 and 55?
- **c.** What is the probability the random variable will assume a value between 40 and 60?

10. Draw a graph for the standard normal distribution. Label the horizontal axis at values of -3, -2, -1, 0, 1, 2 and 3. Then use the table of probabilities for the standard normal distribution to compute the following probabilities.

- **a.** $P(0 \leq Z \leq 1)$.
- **b.** $P(0 \leq Z \leq 1.5)$.
- **c.** $P(0 < Z < 2)$.
- **d.** $P(0 < Z < 2.5)$.

11. Given that Z is a standard normal random variable, compute the following probabilities.

- **a.** $P(-1 \leq Z \leq 0)$.
- **b.** $P(-1.5 \leq Z \leq 0)$.
- **c.** $P(-2 < Z < 0)$.
- **d.** $P(-2.5 \leq Z \leq 0)$.
- **e.** $P(-3 \leq Z \leq 0)$.

12. Given that Z is a standard normal random variable, compute the following probabilities.

COMPLETE SOLUTIONS

a. $P(0 \leq Z \leq 0.83)$.
b. $P(-1.57 \leq Z \leq 0)$.
c. $P(Z > 0.44)$.
d. $P(Z \geq -0.23)$.
e. $P(Z < 1.20)$.
f. $P(Z \leq -0.71)$.

13. Given that Z is a standard normal random variable, compute the following probabilities.

a. $P(-1.98 \leq Z \leq 0.49)$.
b. $P(0.52 \leq Z \leq 1.22)$.
c. $P(-1.75 \leq Z \leq -1.04)$.

14. Given that Z is a standard normal random variable, find z for each situation.

a. The area between 0 and z is 0.4750.
b. The area between 0 and z is 0.2291.
c. The area to the right of z is 0.1314.
d. The area to the left of z is 0.6700.

15. Given that Z is a standard normal random variable, find z for each situation.

a. The area to the left of z is 0.2119.
b. The area between $-z$ and z is 0.9030.
c. The area between $-z$ and z is 0.2052.
d. The area to the left of z is 0.9948.
e. The area to the right of z is 0.6915.

16. Given that Z is a standard normal random variable, find z for each situation.

COMPLETE SOLUTIONS

a. The area to the right of z is 0.01.
b. The area to the right of z is 0.025.
c. The area to the right of z is 0.05.
d. The area to the right of z is 0.10.

Applications

17. Attendance at a rock concert is normally distributed with a mean of 28 000 persons and a standard deviation of 4000 persons. What is the probability, that:

a. more than 28 000 persons will attend?
b. less than 14 000 persons will attend?
c. between 17 000 and 25 000 persons will attend?
d. Suppose the number who actually attended was X and the probability of achieving this level of attendance or higher was found to be 5 per cent. What is X?

18. The holdings of clients of a successful online stockbroker are normally distributed with a mean of £20 000 and standard deviation of £1500. To increase its business, the stockbroker is looking to email special promotions to the top 20 per cent of its clientele based on the value of their holdings. What is the minimum holding of this group?

19. A company has been involved in developing a new pesticide. Tests show that the average proportion, p, of insects killed by administration of x units of the insecticide is given by $p = P(X \leq x)$ where the probability $P(X \leq x)$ relates to a normal distribution with unknown mean and standard deviation.

a. Given that $x = 10$ when $p = 0.4$ and that $x = 15$ when $p = 0.9$, determine the dose that will be lethal to 50 per cent of the insect population on average.
b. If a dose of 17.5 units is administered to each of 100 insects, how many will be expected to die?

6.3 NORMAL APPROXIMATION OF BINOMIAL PROBABILITIES

In Chapter 5, Section 5.4 we presented the discrete binomial distribution. Recall that a binomial experiment consists of a sequence of n identical independent trials with each trial having two possible outcomes: a success or a failure. The probability of a success on a trial is the same for all trials and is denoted by π (Greek pi). The binomial random variable is the number of successes in the n trials, and probability questions pertain to the probability of x successes in the n trials. When the number of trials becomes large, evaluating the binomial probability function by hand or with a calculator is difficult. In addition, the binomial tables in Appendix B do not include values of n greater than 20. Hence, when we encounter a binomial distribution problem with a large number of trials, we may want to approximate the binomial distribution. In cases where the number of trials is greater than 20, $n\pi \geq 5$, and $n(1 - \pi) \geq 5$, the normal distribution provides an easy-to-use approximation of binomial probabilities.

When using the normal approximation to the binomial, we set $\mu = n\pi$ and $\sigma = \sqrt{n\pi(1-\pi)}$ in the definition of the normal curve. Let us illustrate the normal approximation to the binomial by supposing that a particular company has a history of making errors in 10 per cent of its invoices. A sample of 100 invoices has been taken, and we want to compute the probability that 12 invoices contain errors. That is, we want to find the binomial probability of 12 successes in 100 trials.

In applying the normal approximation to the binomial, we set $\mu = n\pi = 100 \times 0.1 = 10$ and $\sigma = \sqrt{n\pi(1-\pi)} = \sqrt{100 \times 0.1 \times 0.9} = 3$. A normal distribution with $\mu = 10$ and $\sigma = 3$ is shown in Figure 6.8.

Recall that, with a continuous probability distribution, probabilities are computed as areas under the probability density function. As a result, the probability of any single value for the random variable is zero. Thus to approximate the binomial probability of 12 successes, we must compute the area under the corresponding normal curve between 11.5 and 12.5. The 0.5 that we add and subtract from 12 is called a **continuity correction factor**. It is introduced because a continuous distribution is being used to approximate a discrete distribution. Thus, $P(X = 12)$ for the *discrete* binomial distribution is approximated by $P(11.5 \leq X \leq 12.5)$ for the *continuous* normal distribution.

Converting to the standard normal distribution to compute $P(11.5 \leq X \leq 12.5)$, we have:

$$z = \frac{x-\mu}{\sigma} = \frac{12.5-10.0}{3} = 0.83 \quad \text{at} \quad X = 12.5$$

And:

$$z = \frac{x-\mu}{\sigma} = \frac{11.5-10.0}{3} = 0.50 \quad \text{at} \quad X = 11.5$$

FIGURE 6.8
Normal approximation to a binomial probability distribution with $n = 100$ and $\pi = 0.10$ showing the probability of 12 errors

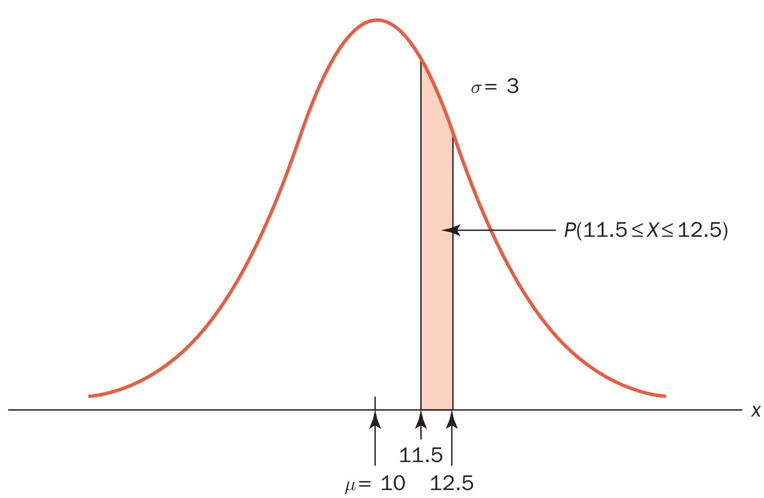
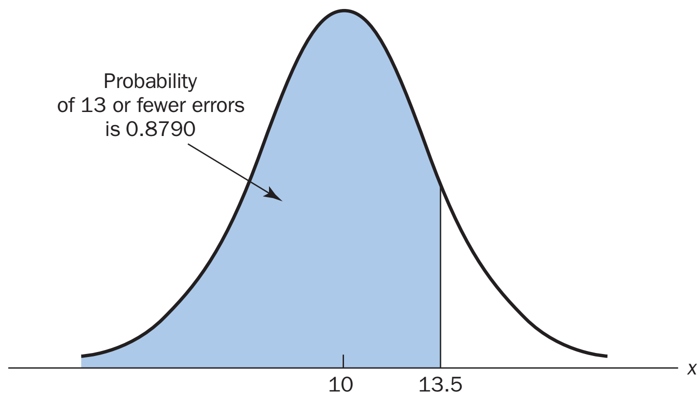


FIGURE 6.9
Normal approximation to a binomial probability distribution with $n = 100$ and $\pi = 0.10$ showing the probability of 13 or fewer errors



Using the standard normal probability table, we find that the area under the curve (in Figure 6.8) to the left of 12.5 is 0.7967. Similarly, the area under the curve to the left of 11.5 is 0.6915. Therefore, the area between 11.5 and 12.5 is $0.7967 - 0.6915 = 0.1052$. The normal approximation to the probability of 12 successes in 100 trials is 0.1052.

For another illustration, suppose we want to compute the probability of 13 or fewer errors in the sample of 100 invoices. Figure 6.9 shows the area under the normal curve that approximates this probability. Note that the use of the continuity correction factor results in the value of 13.5 being used to compute the desired probability. The z value corresponding to $x = 13.5$ is:

$$z = \frac{13.5 - 10.0}{3} = 1.17$$

The standard normal probability table shows that the area under the standard normal curve to the left of 1.17 is 0.8790. The area under the normal curve approximating the probability of 13 or fewer errors is given by the heavily shaded portion of the graph in Figure 6.9.

EXERCISES

Methods

20. A binomial probability distribution has $\pi = 0.20$ and $n = 100$.

- **a.** What is the mean and standard deviation?
- **b.** Is this a situation in which binomial probabilities can be approximated by the normal probability distribution? Explain.
- **c.** What is the probability of exactly 24 successes?
- **d.** What is the probability of 18 to 22 successes?
- **e.** What is the probability of 15 or fewer successes?

COMPLETE SOLUTIONS

21. Assume a binomial probability distribution has $\pi = 0.60$ and $n = 200$.

- **a.** What is the mean and standard deviation?
- **b.** Is this a situation in which binomial probabilities can be approximated by the normal probability distribution? Explain.
- **c.** What is the probability of 100 to 110 successes?
- **d.** What is the probability of 130 or more successes?
- **e.** What is the advantage of using the normal probability distribution to approximate the binomial probabilities? Use part (d) to explain the advantage.

Applications

22. A hotel in Nice has 120 rooms. In the spring months, hotel room occupancy is approximately 75 per cent.

a. What is the probability that at least half of the rooms are occupied on a given day?
b. What is the probability that 100 or more rooms are occupied on a given day?
c. What is the probability that 80 or fewer rooms are occupied on a given day?

6.4 EXPONENTIAL PROBABILITY DISTRIBUTION

The **exponential probability distribution** may be used for random variables such as the time between arrivals at a car wash, the time required to load a truck, the distance between major defects in a highway and so on. The exponential probability density function follows.

Exponential probability density function

$$f(x) = \frac{1}{\mu} e^{-x/\mu} \qquad \text{for } x \geq 0, \mu > 0 \tag{6.4}$$

As an example of the exponential distribution, suppose that X = the time it takes to load a truck at the Schips loading dock follows such a distribution. If the mean, or average, time to load a truck is 15 minutes ($\mu = 15$), the appropriate probability density function is:

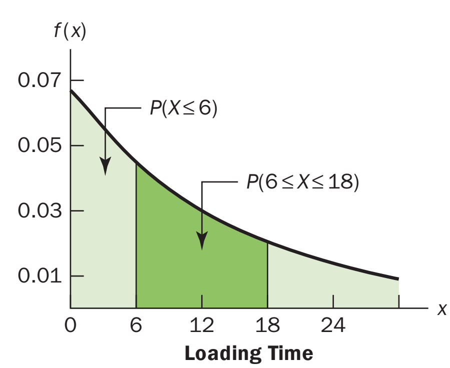
$$f(x) = \frac{1}{15} e^{-x/15}$$

Figure 6.10 is the graph of this probability density function.

Computing probabilities for the exponential distribution

As with any continuous probability distribution, the area under the curve corresponding to an interval provides the probability that the random variable assumes a value in that interval. In the Schips loading dock example, the probability that loading a truck will take six minutes or less ($X \leq 6$) is defined to be the area under the curve in Figure 6.10 from $x = 0$ to $x = 6$.

FIGURE 6.10
Exponential distribution for the Schips loading dock example



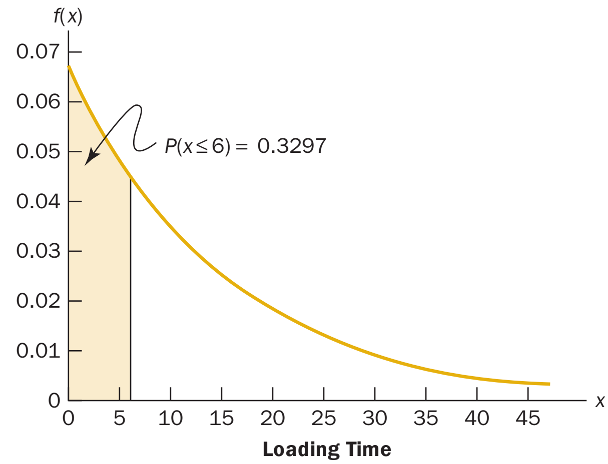


FIGURE 6.11
Probability of a loading time of six minutes of less

Similarly, the probability that loading a truck will take 18 minutes or less ($X \leq 18$) is the area under the curve from $x = 0$ to $x = 18$. Note also that the probability that loading a truck will take between six minutes and 18 minutes ($6 \leq X \leq 18$) is given by the area under the curve from $x = 6$ to $x = 18$.

To compute exponential probabilities such as those just described, we use the following formula (equation (6.5)). It provides the cumulative probability of obtaining a value for the exponential random variable of less than or equal to some specific value denoted by x_0.

Exponential distribution: cumulative probabilities

$$P(X \leq x_0) = 1 - e^{-x_0/\mu} \qquad \textbf{(6.5)}$$

For the Schips loading dock example, X = loading time and $\mu = 15$, which gives us:

$$P(X \leq x_0) = 1 - e^{-x_0\,/\,15}$$

Hence, the probability that loading a truck will take six minutes or less is:

$$P(X \leq 6) = 1 - e^{-6/15} = 0.3297$$

Figure 6.11 shows the area or probability for a loading time of six minutes or less. Using equation (6.5), we calculate the probability of loading a truck in 18 minutes or less:

$$P(X \leq 18) = 1 - e^{-18/15} = 0.6988$$

Thus, the probability that loading a truck will take between six minutes and 18 minutes is equal to $0.6988 - 0.3297 = 0.3691$. Probabilities for any other interval can be computed similarly.

In the preceding example, the mean time it takes to load a truck is $\mu = 15$ minutes. A property of the exponential distribution is that the mean of the distribution and the standard deviation of the distribution are *equal*. Thus, the standard deviation for the time it takes to load a truck is $\sigma = 15$ minutes. The variance is $\sigma^2 = (15)^2 = 225$.

Relationship between the Poisson and exponential distributions

In Chapter 5, Section 5.5 we introduced the Poisson distribution as a discrete probability distribution that is often useful in examining the number of occurrences of an event over a specified interval of time or space. Recall that the Poisson probability function is:

$$p(x) = \frac{\mu^x e^{-\mu}}{x!}$$

where:

μ = expected value or mean number of occurrences over a specified interval.

The continuous exponential probability distribution is related to the discrete Poisson distribution. If the Poisson distribution provides an appropriate description of the number of occurrences per interval, the exponential distribution provides a description of the length of the interval between occurrences.

To illustrate this relationship, suppose the number of cars that arrive at a car wash during one hour is described by a Poisson probability distribution with a mean of ten cars per hour. The Poisson probability function that gives the probability of X arrivals per hour is:

$$p(x) = \frac{10^x e^{-10}}{x!}$$

Because the average number of arrivals is ten cars per hour, the average time between cars arriving is:

$$\frac{1 \text{ hour}}{10 \text{ cars}} = 0.1 \text{ hour/car}$$

Thus, the corresponding exponential distribution that describes the time between the arrivals has a mean of $\mu = 0.1$ hour per car; as a result, the appropriate exponential probability density function is:

$$f(x) = \frac{1}{0.1} e^{-x/0.1} = 10e^{-10x}$$

EXERCISES

Methods

23. Consider the following exponential probability density function.

$$f(x) = \frac{1}{8} e^{-x/2} \quad \text{for } x \geq 0$$

a. Find $P(X \leq 6)$.
b. Find $P(X \leq 4)$.
c. Find $P(X \geq 6)$.
d. Find $P(4 \leq X \leq 6)$.

24. Consider the following exponential probability density function.

$$f(x) = \frac{1}{3} e^{-x/3} \quad \text{for } x \geq 0$$

a. Write the formula for $P(X \leq x_0)$.
b. Find $P(X \leq 2)$.
c. Find $P(X \geq 3)$.
d. Find $P(X \leq 5)$.
e. Find $P(2 \leq X \leq 5)$.

Applications

25. In a parts store in Mumbai, customers arrive randomly. The cashier's service time is random but it is estimated it takes an average of 30 seconds to serve each customer.

a. What is the probability a customer must wait more than two minutes for service?
b. Suppose average service time is reduced to 25 seconds. How does this affect the calculation for (a) above?

26. The time between arrivals of vehicles at a particular intersection follows an exponential probability distribution with a mean of 12 seconds.

a. Sketch this exponential probability distribution.
b. What is the probability that the arrival time between vehicles is 12 seconds or less?
c. What is the probability that the arrival time between vehicles is six seconds or less?
d. What is the probability of 30 or more seconds between vehicle arrivals?

COMPLETE SOLUTIONS

27. According to Barron's 1998 Primary Reader Survey, the average annual number of investment transactions for a subscriber is 30 (www.barronsmag.com, 28 July 2000). Suppose the number of transactions in a year follows the Poisson probability distribution.

a. Show the probability distribution for the time between investment transactions.
b. What is the probability of no transactions during the month of January for a particular subscriber?
c. What is the probability that the next transaction will occur within the next half month for a particular subscriber?

ONLINE RESOURCES

For the data files, additional online summary, questions, answers and the software section for this chapter, go to the online platform.

SUMMARY

This chapter extended the discussion of probability distributions to the case of continuous random variables. The major conceptual difference between discrete and continuous probability distributions involves the method of computing probabilities. With discrete distributions, the probability function $p(x)$ provides the probability that the random variable X assumes various values. With continuous distributions, the probability density function $f(x)$ does not provide probability values directly. Instead, probabilities are given by areas under the curve or graph of $f(x)$. Three continuous probability distributions – the uniform, normal and exponential distributions were the particular focus – with detailed examples showing how probabilities could be straightforwardly computed. In addition, relationships between the binomial and normal distributions and Poisson and exponential distribution were established and related probability results, exploited.

KEY TERMS

Continuity correction factor
Exponential probability distribution
Normal probability distribution
Probability density function
Standard normal probability distribution
Uniform probability distribution

KEY FORMULAE

Uniform Probability Density Function

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \leq X \leq b \\ 0 & \text{elsewhere} \end{cases} \tag{6.1}$$

Normal Probability Density Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)2/2\sigma^2} \tag{6.2}$$

Converting to the Standard Normal Distribution

$$Z = \frac{X - \mu}{\sigma} \tag{6.3}$$

Exponential Probability Density Function

$$f(x) = \frac{1}{\mu} e^{-x/\mu} \quad \text{for } x \geq 0, \mu > 0 \tag{6.4}$$

Exponential Distribution: Cumulative Probabilities

$$p(X \leq x_0) = 1 - e^{-x_0/\mu} \tag{6.5}$$

CASE PROBLEM 1

Prix-Fischer Toys

Prix-Fischer Toys sells a variety of new and innovative children's toys. Management learned that the pre-holiday season is the best time to introduce a new toy, because many families use this time to look for new ideas for December holiday gifts. When Prix-Fischer discovers a new toy with good market potential, it chooses an October market entry date.

In order to get toys in its stores by October, Prix-Fischer places one-time orders with its manufacturers in June or July of each year. Demand for children's toys can be highly volatile. If a new toy catches on, a sense of shortage in the market place often increases the demand to high levels and large profits can be realized. However, new toys can also flop, leaving Prix-Fischer stuck with high levels of inventory that must be sold at reduced prices. The most important question the company faces is deciding how many units of a new toy should be purchased to meet anticipated sales demand. If too few are purchased, sales will be lost; if too many are purchased, profits will be reduced because of low prices realized in clearance sales.

For the coming season, Prix-Fischer plans to introduce a new talking bear product called Chattiest Teddy. As usual, Prix-Fischer faces the decision of how many Chattiest Teddy units to order for the coming holiday season. Members of the management team suggested order quantities of 15 000, 18 000, 24 000 or 28 000 units. The wide range of order quantities suggested, indicate considerable disagreement concerning the market potential. The product management team asks you for an analysis of the stock-out probabilities for various order quantities, an estimate of the profit potential, and to help make an order quantity recommendation.

Prix-Fischer expects to sell Chattiest Teddy for €24 based on a cost of €16 per unit. If inventory remains after the holiday season, Prix-Fischer will sell all surplus inventory for €5 per unit. After reviewing the sales history of similar products, Prix-Fischer's senior sales forecaster predicted an expected demand of 20 000 units with a 0.90 probability that demand would be between 10 000 units and 30 000 units.

Managerial report

Prepare a managerial report that addresses the following issues and recommends an order quantity for the Chattiest Teddy product.

1. Use the sales forecaster's prediction to describe a normal probability distribution that can be used to approximate the demand distribution. Sketch the distribution and show its mean and standard deviation.
2. Compute the probability of a stock-out for the order quantities suggested by members of the management team.
3. Compute the projected profit for the order quantities suggested by the management team under three scenarios: worst case in which sales = 10 000 units, most likely case in which sales = 20 000 units, and best case in which sales = 30 000 units.
4. One of Prix-Fischer's managers felt that the profit potential was so great that the order quantity should have a 70 per cent chance of meeting demand and only a 30 per cent chance of any stock-outs. What quantity would be ordered under this policy, and what is the projected profit under the three sales scenarios?
5. Provide your own recommendation for an order quantity and note the associated profit projections. Provide a rationale for your recommendation.

CASE PROBLEM 2

Queuing patterns in a retail furniture store

The assistant manager of one of the larger stores in a retail chain selling furniture and household appliances has recently become interested in using quantitative techniques in the store operation. To help resolve a longstanding queuing problem, data have been collected on the time between customer arrivals and the time that a given number of customers were in a particular store department. Relevant details are summarized in Tables 6.1 and 6.2 respectively. Corresponding data on service times per customer are tabulated in Table 6.3.

In order to arrive at an appropriate solution strategy for the department's queuing difficulties, the manager has come to you for advice on possible statistical patterns that might apply to this information.

1. By plotting the arrival and service patterns shown in Tables 6.1 and 6.3, show that they can each be reasonably represented by an exponential distribution.

TABLE 6.1 Time between arrivals (during a four-hour period)

Time between arrivals (in minutes)	Frequency
0.0 < 0.2	31
0.2 < 0.4	32
0.4 < 0.6	23
0.6 < 0.8	21
0.8 < 1.0	19
1.0 < 1.2	11
1.2 < 1.4	14
1.4 < 1.6	8
1.6 < 1.8	6
1.8 < 2.0	9
2.0 < 2.2	6
2.2 < 2.4	4
2.4 < 2.6	5
2.6 < 2.8	4
2.8 < 3.0	4
3.0 < 3.2	3
More than 3.2	10

TABLE 6.2 Time that n customers were in the department (during a four-hour period)

Number of customers, n	Time (in minutes)
0	16.8
1	35.5
2	52
3	49
4	29.3
5	18.8
6	13.6
7	9.6
8	5.8
More than 8	9.6

TABLE 6.3 Service time (from a sample of 31)

Service time (in minutes)	Frequency
Less than 1	5
1 < 2	7
2 < 3	6
3 < 4	4
4 < 5	2
5 < 6	3
6 < 7	1
7 < 8	1
8 < 9	1
9 < 10	0
More than 10	1

2. If λ = mean arrival rate for the queuing system here and μ, the mean service rate for each channel, estimate the mean arrival time $(1/\lambda)$ and mean service time $(1/\mu)$ respectively.

3. If k = the number of service channels and the mean service time for the system (store) is greater than the mean arrival rate (i.e. $k\mu > \lambda$) then the following formulae can be shown to apply to the system 'in the steady state' subject to certain additional mathematical assumptions.[1]

   a. The probability there are no customers in the system

$$p(0) = \frac{1}{\sum_{n-0}^{k-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^k}{k!} \frac{k\mu}{(k\mu-\lambda)}}$$

   b. The average number of customers in the queue

$$L_q = \frac{(\lambda/\mu)^k \lambda\mu}{(k-1)!(k\mu-\lambda)^2} p(0)$$

   c. The average number of customers in the store

$$L = L_q + \frac{\lambda}{\mu}$$

   d. The average time a customer spends in the queue

$$w_q = \frac{L_q}{\lambda}$$

[1] The queue has two or more channels; the mean service rate μ is the same for each channel; arrivals wait in a single queue and then move to the first open channel for service; the queue discipline is first-come, first-served (FCFS).

e. The average time a customer spends in the store

$$W = W_q + \frac{1}{\mu}$$

f. The probability of n customers in the system

$$p(n) = \frac{(\lambda + \mu)^n}{n!} p(0) \quad \text{for } n\ k$$

$$p(n) = \frac{(\lambda + \mu)^n}{k!k^{(n-k)}} p(0) \quad \text{for } n > k$$

According to this model, what is the smallest value that k can take? If this is the number of channels that the retailer currently operates, estimate the above operating characteristics for the store. How would these values change if the k channels were increased by one or two extra channels? Discuss what factors might influence the retailer in arriving at an appropriate value of k.